Supplement 3: The Lorentz Group and The Poincaré Group

Having introduced the $SO(N)$ and the $O(N)$ groups in Supplement 2, it is now straightforward to discuss the basic properties of the Lorentz and Poincaré groups, which are important to the studies of relativistic quantum field theory.

• The Lorentz Group

The Lorentz group is an $SO(3,1)$ group analogous to the $SO(4)$ group, a 4-dimensional rotation group. There are 6 generators in the Lorentz group, three of them related to the boosts and the other three related to the three-dimensional rotation. Specifically, the Lorentz group is defined by the set of all (4×4) real matrices that leave the following quantity invariant:

$$s^2 = c^2t^2 - x^i x^i = \left(x^0\right)^2 - \left(x^i\right)^2 = x^\mu g_{\mu\nu} x^\nu , \tag{S3.1}$$

where s denotes the interval between two events in spacetime, $g_{\mu\nu}$ is the metric of the Lorentz group, i = 1, 2, 3 for the three spatial coordinates, and μ = 0, 1, 2, 3 for the four-dimensional spacetime coordinates. In addition, we have adopted the summation convention for repeating indices, and the following convention for the *contravariant* and *covariant* 4-vectors x^μ and x_μ, respectively:

$$x^\mu = \left(x^0, x^1, x^2, x^3\right) = \left(ct, x, y, z\right),$$
$$x_\mu = \left(x_0, x_1, x_2, x_3\right) = \left(ct, -x, -y, -z\right).$$

The minus sign in EQ. (S3.1) distinguishes the $SO(3,1)$ Lorentz group from the group $SO(4)$. [In general, an orthogonal group that preserves a metric with M indices of one sign and N indices of another sign is denoted as $O(M,N)$.]

Inserting the Lorentz transformation $x'^\nu = \Lambda^\nu{}_\mu \, x^\mu$ into the invariant, we find that the Λ matrices must satisfy the relation:

$$g_{\mu\nu} = \Lambda_\mu{}^\rho \, g_{\rho\sigma} \, \Lambda^\sigma{}_\nu . \tag{S3.2}$$

In the Minkowski space, we replace $g_{\mu\nu}$ by $\eta_{\mu\nu}$ and the latter is given by:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{S3.3}$$

Given that there are 6 generators in the Lorentz group, we expect to find 6 independent Λ matrices. Before determining the Λ matrices explicitly, we note that the Λ matrices must preserve the length x in spacetime, so that

$$x'^\mu x'_\mu = x^\mu x_\mu \quad \rightarrow \quad \left(\Lambda^\mu{}_\rho x^\rho\right)\left(\Lambda_\mu{}^\sigma x_\sigma\right) = \Lambda^\mu{}_\rho \Lambda_\mu{}^\sigma x^\rho x_\sigma = x^\mu x_\mu \quad \leftrightarrow \quad \Lambda^\mu{}_\rho \Lambda_\mu{}^\sigma = \delta_\rho^\sigma . \tag{S3.4}$$

From EQ. (S3.4), we find that the inverse of Λ is given by $\left(\Lambda^{-1}\right)^\mu{}_\nu = \Lambda_\nu{}^\mu$.

Next, we introduce the operator $L^{\mu\nu}$ in order to define the action of the Lorentz group on fields:

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right). \tag{S3.5}$$

This definition generates the algebra of the Lorentz group:

$$\left[L^{\mu\nu}, L^{\rho\sigma}\right] = i\left(\eta^{\nu\rho} L^{\mu\sigma} - \eta^{\mu\rho} L^{\nu\sigma} + \eta^{\mu\sigma} L^{\nu\rho} - \eta^{\nu\sigma} L^{\mu\rho}\right), \tag{S3.6}$$

and it has essentially the same structure as that given in EQ. (S2.54). The operators $L^{\mu\nu}$ can be considered as the generators for tensors. Similar to the case of spinor representations in $SO(N)$, we may define

$$U(\Lambda) = \exp\left(-i\varepsilon^{\mu\nu} L_{\mu\nu}\right), \tag{S3.7}$$

so that the action of the Lorentz group on a vector field A^μ can be expressed as:

$$U(\Lambda) A^\mu(x) U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu A^\nu(x'), \tag{S3.8}$$

where we have used the fact that $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \varepsilon^{\mu\nu} + \cdots$ and the following identity:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A,[A, B]] + \frac{1}{3!}[A,[A,[A, B]]] + \ldots \tag{S3.9}$$

Comparing EQ. (S2.57) for the spinor representations in an $SO(N)$ group with EQ. (S3.8) for the Lorentz group, we can say that the objects Γ^μ transform like vectors in the Lorentz group. On the other hand, a scalar $\phi(x)$ in a Lorentz group transforms according to the following:

$$U(\Lambda)\phi(x) U^{-1}(\Lambda) = \phi(x'). \tag{S3.10}$$

To parameterize 6 independent Λ^μ_ν explicitly, we consider the following Lorentz transformation:

$$x' = \frac{x + vt}{\sqrt{1-v^2}} \equiv x\cosh\varphi + t\sinh\varphi\,;\ y' = y\,;\ z' = z\,;\ t' = \frac{t + vx}{\sqrt{1-v^2}} \equiv t\cosh\varphi + x\sinh\varphi\,;$$

so that

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{S3.11}$$

For sufficiently small φ, the transformation becomes

$$x' = x + t\varphi\ ;\ \ y' = y\ ;\ \ z' = z\ ;\ \ t' = t + x\varphi\,.$$

Consequently, we can define three hermitian matrices K_i (i = 1, 2, 3) for the Lorentz boosts along x, y, z:

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{S3.12}$$

For a Lorentz boost from the rest frame to finite velocity along the x-axis, the Lorentz transformation is:

$$\exp\left(-i\,\boldsymbol{\varphi K}\right)=\exp\left(-i\,\varphi K_1\right)=\begin{pmatrix} I\cosh\varphi & 0 \\ 0 & I \end{pmatrix}-\left(i\sinh\varphi\right)K_1, \tag{S3.13}$$

where I is a (2×2) unit matrix.

Besides the three matrices representing the Lorentz boosts, the other three generators are associated with the three-dimensional rotation J_i (i = 1, 2, 3):

$$J_1=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},\qquad J_2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},\qquad J_3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{S3.14}$$

Similar to the Lorentz boost discussed above, the rotation by θ relative to the x-axis can be expressed as

$$\exp\left(-i\,\boldsymbol{\theta J}\right)=\exp\left(-i\,\theta J_1\right)=\begin{pmatrix} I & 0 \\ 0 & I\cos\theta \end{pmatrix}-\left(i\sin\theta\right)J_1. \tag{S3.15}$$

If we restrict to a two-dimensional rotation, the $SO(2)$ group, we'll recover the familiar (2×2) rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ from the above expression by removing the row and column associated I.

The above 6 independent generators defined in EQs. (S3.12) and (S3.14) satisfy the following algebra:

$$\left[J_i,J_j\right]=i\varepsilon_{ijk}J_k,\qquad \left[J_i,K_j\right]=i\varepsilon_{ijk}K_k,\qquad \left[K_i,K_j\right]=-i\varepsilon_{ijk}J_k. \tag{S3.16}$$

From EQ. (S3.16), it is clear that pure Lorentz transformations associated with the K-generators do not form a group. We also note that the J- and K-generators of the Lorentz group are related to the general expression for $M^{\mu\nu}$ in EQ. (S2.54) via the following definitions:

$$K^i\equiv M^{0i},\qquad J^i\equiv\frac{1}{2}\varepsilon^{ijk}M_{jk}. \tag{S3.17}$$

Consequently, the matrices $\Lambda^{\mu}{}_{\nu}$ in the Lorentz group become (μ, ν = 0, 1, 2, 3):

$$\Lambda=\exp\left(-i\,\theta_{\mu\nu}M^{\mu\nu}\right). \tag{S3.18}$$

Next, we consider the spinor representations in the Lorentz group. Let us define the generators **A** and **B** using **J** and **K** as follows:

$$\mathbf{A}=\frac{1}{2}\left(\mathbf{J}+i\mathbf{K}\right),\qquad \mathbf{B}=\frac{1}{2}\left(\mathbf{J}-i\mathbf{K}\right). \tag{S3.19}$$

From EQ. (S3.16), we find:

$$\left[A_i,A_j\right]=i\varepsilon_{ijk}A_k,\qquad \left[B_i,B_j\right]=i\varepsilon_{ijk}B_k,\qquad \left[A_i,B_j\right]=0. \tag{S3.20}$$

Equation (S3.20) shows that **A** and **B** each generate a group $SU(2)$, and the two groups commute. Therefore the Lorentz group can be expressed as $SO(3,1)\sim SU(2)\otimes SU(2)$, as stated before. In this context, we may describe the transformation of states in terms of two angular momenta (j, j'), where j corresponds to **A** and j' corresponds to **B**. From EQ. (S3.20), we can identify special cases of (j,0) and (0, j') with the conditions **J** =

$i\mathbf{K}$ and $\mathbf{J} = -i\mathbf{K}$, respectively. These two special cases correspond to spin zero in one of the states. Now we may define two types of spinor as follows:

$$\xi \equiv \left(\frac{1}{2}, 0\right): \qquad \mathbf{J}^{(1/2)} = \boldsymbol{\sigma}/2, \quad \mathbf{K}^{(1/2)} = -i\boldsymbol{\sigma}/2, \tag{S3.21}$$

$$\eta \equiv \left(0, \frac{1}{2}\right): \qquad \overline{\mathbf{J}}^{(1/2)} = \boldsymbol{\sigma}/2, \quad \overline{\mathbf{K}}^{(1/2)} = i\boldsymbol{\sigma}/2. \tag{S3.22}$$

If $(\boldsymbol{\theta}, \boldsymbol{\varphi})$ represent the parameters of a rotation and a Lorentz boost, we find that the spinors ξ and η transform according to the following:

$$\xi \to \exp\left(-i\,\mathbf{J}^{(1/2)} \bullet \boldsymbol{\theta} - i\,\mathbf{K}^{(1/2)} \bullet \boldsymbol{\varphi}\right)\xi = \exp\left[-i\frac{\boldsymbol{\sigma}}{2} \bullet (\boldsymbol{\theta} + i\boldsymbol{\varphi})\right]\xi \equiv \mathbf{M}\xi\,, \tag{S3.23}$$

$$\eta \to \exp\left(-i\,\overline{\mathbf{J}}^{(1/2)} \bullet \boldsymbol{\theta} - i\,\overline{\mathbf{K}}^{(1/2)} \bullet \boldsymbol{\varphi}\right)\eta = \exp\left[-i\frac{\boldsymbol{\sigma}}{2} \bullet (\boldsymbol{\theta} - i\boldsymbol{\varphi})\right]\eta \equiv \mathbf{N}\eta\,. \tag{S3.24}$$

The representations **M** and **N** of the Lorentz group are nonequivalent so that they cannot be transformed into each other through a similarity transformation $\mathbf{N} = \mathbf{SMS}^{-1}$ with **S** being a matrix. In fact, **M** and **N** are (2 × 2) complex matrices with unit determinant and related by the Pauli matrix $\boldsymbol{\sigma}_2$:

$$\mathbf{N} = \boldsymbol{\zeta}\mathbf{M}^{*}\boldsymbol{\zeta}^{-1} \qquad \text{with } \boldsymbol{\zeta} = -i\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{S3.25}$$

These (2 × 2) complex matrices with unit determinant actually form a group known as *SL*(2,*C*), and they constitute the spinor irreducible representations.

If we introduce parity operation to the spinor representations, we find that the generators **K** change sign because they behave like vectors while the generators **J** do not because they behave like axial vectors or pseudo-vectors. Hence, under parity operation, the representations (*j*,0) and (0,*j*) become interchanged according to EQs. (S3.21) and (S3.22). Consequently, we have $\xi \leftrightarrow \eta$ under parity operation, which implies that the Lorentz group extended by parity can no longer be described in terms of the 2-spinor representations. Rather, we must introduce the 4-spinor representation:

$$\psi \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{S3.26}$$

and the 4-spinor representation transforms under the generalized Lorentz transformation $(\boldsymbol{\theta}, \boldsymbol{\varphi})$ according to the following:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \to \begin{pmatrix} e^{-i\frac{\boldsymbol{\sigma}}{2} \bullet (\boldsymbol{\theta} + i\boldsymbol{\varphi})} & 0 \\ 0 & e^{-i\frac{\boldsymbol{\sigma}}{2} \bullet (\boldsymbol{\theta} - i\boldsymbol{\varphi})} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \equiv \begin{pmatrix} D(\Lambda) & 0 \\ 0 & \overline{D}(\Lambda) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{S3.27}$$

where

$$\overline{D}(\Lambda) = \boldsymbol{\zeta} D^{*}(\Lambda) \boldsymbol{\zeta}^{-1}, \tag{S3.28}$$

and Λ denotes the Lorentz transformation $x'^{\nu} = \Lambda^{\nu}{}_{\mu}\, x^{\mu}$ according to the definition in EQ. (S3.2).

Now let's consider how the 4-spinor representation transforms under a pure Lorentz boost. We'll show that the celebrated Dirac equation can in fact be derived from such consideration. For a pure Lorentz boost, we take $\boldsymbol{\theta} = 0$ and rewrite the 2-spinors ξ and η as follows:

$$\xi \to \phi_R, \qquad \eta \to \phi_L, \tag{S3.29}$$

where we use R and L to represent the right and left polarizations. Under a pure Lorentz boost along a direction denoted by a unit vector $\hat{p}$, we obtain

$$\begin{aligned}
\phi_R(\mathbf{p}) &= e^{\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}}\phi_R(0) = \left[\cosh\left(\frac{\varphi}{2}\right) + \sinh\left(\frac{\varphi}{2}\right)(\boldsymbol{\sigma}\cdot\hat{p})\right]\phi_R(0), \\
&= \left[\left(\frac{\gamma+1}{2}\right)^{1/2} + \left(\frac{\gamma-1}{2}\right)^{1/2}(\boldsymbol{\sigma}\cdot\hat{p})\right]\phi_R(0), \qquad \left(\gamma \equiv \cosh\varphi = \frac{E}{m}, \quad \gamma\beta \equiv \sinh\varphi\right) \\
&= \frac{(E+m) + \boldsymbol{\sigma}\cdot\mathbf{p}}{\left[2m(E+m)\right]^{1/2}}\phi_R(0),
\end{aligned} \tag{S3.30}$$

$$\begin{aligned}
\phi_L(\mathbf{p}) &= e^{-\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varphi}}\phi_R(0) = \left[\cosh\left(\frac{\varphi}{2}\right) - \sinh\left(\frac{\varphi}{2}\right)(\boldsymbol{\sigma}\cdot\hat{p})\right]\phi_L(0) \\
&= \frac{(E+m) - \boldsymbol{\sigma}\cdot\mathbf{p}}{\left[2m(E+m)\right]^{1/2}}\phi_L(0),
\end{aligned} \tag{S3.31}$$

where we have used the identity $\gamma\beta m\hat{p} = \mathbf{p}$. We note that $\phi_R(0) = \phi_L(0)$ because when a particle at rest, we cannot define whether its spin is left- or right-handed, and we have taken $c = 1$ and the total energy of the particle E. Thus, EQs. (S3.20) and (S3.31) lead to the following relations:

$$\phi_R(\mathbf{p}) = \frac{(E + \boldsymbol{\sigma}\cdot\mathbf{p})}{m}\phi_L(\mathbf{p}), \tag{S3.32}$$

$$\phi_L(\mathbf{p}) = \frac{(E - \boldsymbol{\sigma}\cdot\mathbf{p})}{m}\phi_R(\mathbf{p}), \tag{S3.33}$$

which can be rearranged into the matrix form:

$$\begin{pmatrix} -m & p_0 + \boldsymbol{\sigma}\cdot\mathbf{p} \\ p_0 - \boldsymbol{\sigma}\cdot\mathbf{p} & -m \end{pmatrix}\begin{pmatrix} \phi_R(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix} = 0, \qquad \text{where } E \equiv p_0. \tag{S3.34}$$

If we further define the momentum-dependent 4-spinor

$$\psi(p) \equiv \begin{pmatrix} \phi_R(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix} \tag{S3.35}$$

and the following 4×4 "γ-matrices":

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad (i = 1,2,3) \tag{S3.36}$$

we find that EQ. (S3.34) becomes:

$$\left(\gamma^0 p_0 + \gamma^i p_i - m\right)\psi(p) = \left(\gamma^\mu p_\mu - m\right)\psi(p) = 0\,. \tag{S3.37}$$

Equation (S3.37) is the celebrated Dirac equation for massive spin ½ particles. On the other hand, for massless particles, the Dirac equation for 4-spinor decouples into two equations for $\phi_R(\mathbf{p})$ and $\phi_L(\mathbf{p})$:

$$\left(p_0 - \boldsymbol{\sigma}\bullet\mathbf{p}\right)\phi_R(\mathbf{p}) = 0, \tag{S3.38}$$

$$\left(p_0 + \boldsymbol{\sigma}\bullet\mathbf{p}\right)\phi_L(\mathbf{p}) = 0. \tag{S3.39}$$

Equations (S3.38) and (S3.39) are known as the Weyl equations, and $\phi_R(\mathbf{p})$ and $\phi_L(\mathbf{p})$ for the massless particles are the Weyl spinors. We shall return to the Dirac equation and spinors in Part II.4. Overall, the tensor and spinor representations of the Lorentz group introduced in this section will become handy in our later discussion of relativistic quantum field theory.

• The Poincaré Group

Generally in quantum mechanics we are only interested in the unitary representations of a symmetry group, because only the unitary representations can preserve the transition probabilities between two eigen-states, as measured in different reference frames. In this context, the fact that the irreducible spinor representation of the Lorentz group is not unitary, as shown in EQ. (S3.24), appears unsatisfactory. This result is due to the fact that the Lorentz group is not compact because the parameter β associated with the Lorentz boost takes on values along an open line from 0 to 1, unlike the rotation group where the angle extends from $\theta = 0$ to 2π. In fact, it was first realized by Wigner that the true symmetry group for particle physics is not the (homogeneous) Lorentz group. Rather, the underlying symmetry group for particle physics must consist of translations in spacetime in addition to the Lorentz boosts and rotations. Such a group is the inhomogeneous Lorentz group, also known as the Poincaré group. Careful analysis of the Poincaré group can lead to insightful understanding of the nature of spin, although we shall not elaborate this subject here. In the following we focus on the definition, Lie algebra and the transformation rules of the Poincaré group, and then remark on the physical invariants of the Poincaré group.

The generator of spacetime translations P_μ gives the transformation:

$$x^\mu \to x'^\mu = x^\mu + a^\mu\,, \tag{S3.40}$$

and

$$P_\mu = i\frac{\partial}{\partial x^\mu}. \tag{S3.41}$$

There are 4 translation generators in the Poincaré group, in addition to 3 generators for Lorentz boosts and 3 generators for rotations. Therefore, there are totally 10 generators in the Poincaré group. The Lie algebra for the Poincaré group includes the commutation relations of the Lorentz group given in EQ. (S3.16), which can be rewritten using EQ. (S3.17) into the following:

$$\left[M_{\mu\nu}, M_{\rho\sigma}\right] = -i\left(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}\right), \qquad (\mu,\nu,\rho,\sigma = 0,1,2,3) \tag{S3.42}$$

where the Lorentz group generators are given by:

$$M_{ij} \equiv \varepsilon_{ijk}J_k = -M_{ji},$$
$$M_{0i} \equiv K_i = -M_{i0} \qquad (i,j,k = 1,2,3). \tag{S3.43}$$

In addition, the commutation relations involving the translation generators are:

$$\left[P_\mu, P_\nu\right] = 0, \tag{S3.44}$$

$$\left[P_\mu, M_{\rho\sigma}\right] = i\left(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho\right). \tag{S3.45}$$

The general inhomogeneous Lorentz transformation that includes the Lorentz boosts, rotations and spacetime translations is given by

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{S3.46}$$

where the matrix Λ that includes boosts and rotations is given in EQ. (S3.18).

We may express the general transformation in the Poincaré group in EQ. (S3.46) by the notation $\{\Lambda,a\}$. The unit element is therefore given by $\{1,0\}$, and the compounded transformation $\left\{\overline{\Lambda},\overline{a}\right\}\left\{\Lambda,a\right\}$ yields:

$$\left\{\overline{\Lambda},\overline{a}\right\}\left\{\Lambda,a\right\} = \left\{\overline{\Lambda}\Lambda, \overline{\Lambda}a + \overline{a}\right\}, \tag{S3.47}$$

because

$$x''^\mu = \overline{\Lambda}^\mu{}_\nu\left(x'^\nu\right) + \overline{a}^\mu = \overline{\Lambda}^\mu{}_\nu\left(\Lambda^\nu{}_\rho\, x^\rho + a^\nu\right) + \overline{a}^\mu = \overline{\Lambda}^\mu{}_\nu\,\Lambda^\nu{}_\rho\, x^\rho + \left(\overline{\Lambda}^\mu{}_\nu\, a^\nu + \overline{a}^\mu\right). \tag{S3.48}$$

Finally, we remark that there are only two invariants in the Poincaré group that commute with all generators, known as the *Casimir invariants* or *Casimir operators*. (The Poincaré group is of rank 2, so that there are only two invariants -- In general, the rank of an $SO(N)$ group is $N/2$ if N is even and is $(N-1)/2$ if N is odd.) The first Casimir invariant C_1 is associated with the mass invariance, and the second Casimir invariant C_2 refers to spin invariance. Specifically,

$$C_1 \equiv P^\mu P_\mu, \tag{S3.40}$$

$$C_2 \equiv W_\mu W^\mu = -m^2 s(s+1), \tag{S3.50}$$

where s is the spin of the particle, W_μ is the Pauli-Lubanski pseudo-vector defined as follows:

$$W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma, \tag{S3.51}$$

and $\varepsilon_{\mu\nu\rho\sigma}$ denotes the totally anti-symmetric symbol in four dimensions.

Further Readings:

1. Schweber, “An Introduction to Relativistic Quantum Field Theory”, Chapter 2.